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Scalable algorithm for Solving Convex Feasibility Problems

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Convex Feasibility Problem

- The problem of finding a point in the intersection of a finite family of closed convex sets in the Euclidean space
- Applications
  - Image reconstruction
  - Quantum information science
  - Asset-liability management
  - Scheduling
  - Algorithmic trading

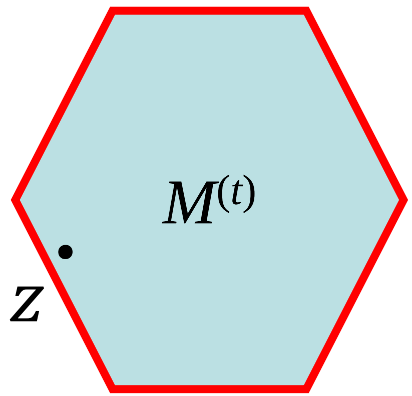
Non-Stationary Linear Feasibility Problem

Very Big System

$$A^{(t)} x \leq b^{(t)}$$

- $x \in \mathbb{R}_n$
- $A^{(t)}$ – matrix $m \times n$
- $b^{(t)}$ – vector of dimension n
- $t \in \mathbb{R}_{\geq 0}$ – time

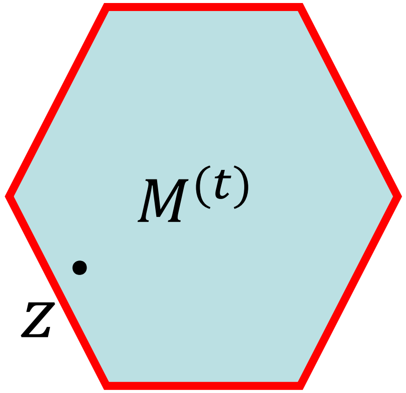
Geometric Interpretation of Linear Feasibility Problem



$$A^{(t)}x \leq b^{(t)} \Leftrightarrow x \in M^{(t)}$$

Troubles with Non-Stationarity

We can't simply solve the system of inequalities since while the calculations are performed the polytope is changing its position in the space!



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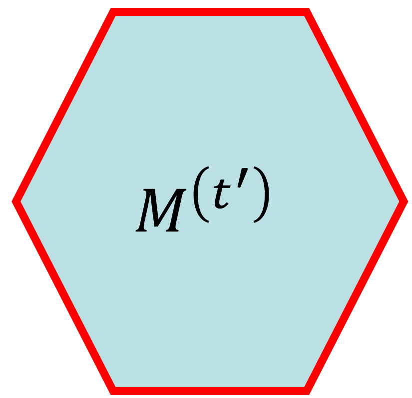


z •

$$A^{(t')}x \leq b^{(t')} \Leftrightarrow x \in M^{(t')}$$

$$A^{(t)}x \leq b^{(t)} \Leftrightarrow x \in M^{(t)}$$

Algorithm for Non-Stationary Linear Feasibility Problem

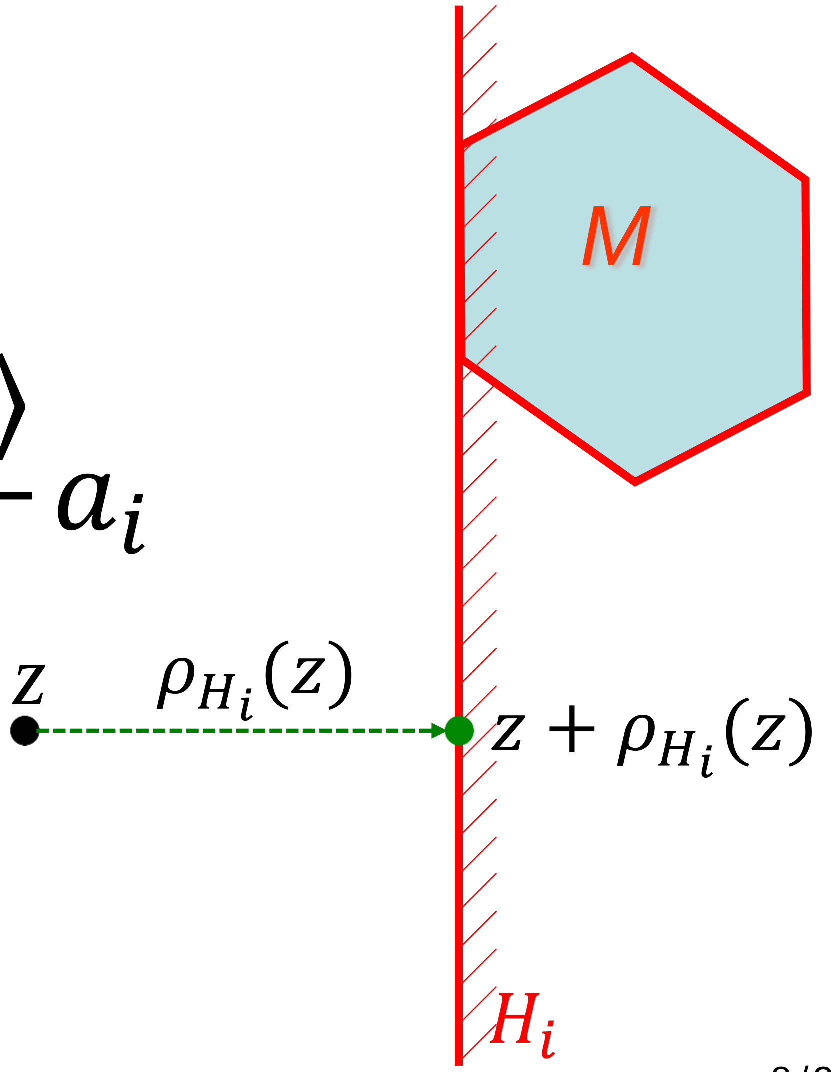
- Requirements:
  - High scalability
  - Self-correcting

- Cimmino algorithm for inequalities
  - Projective
  - Iterative

Vector of Projection onto Hyperplane H_i

$$H_i : \langle a_i, x \rangle = b_i$$

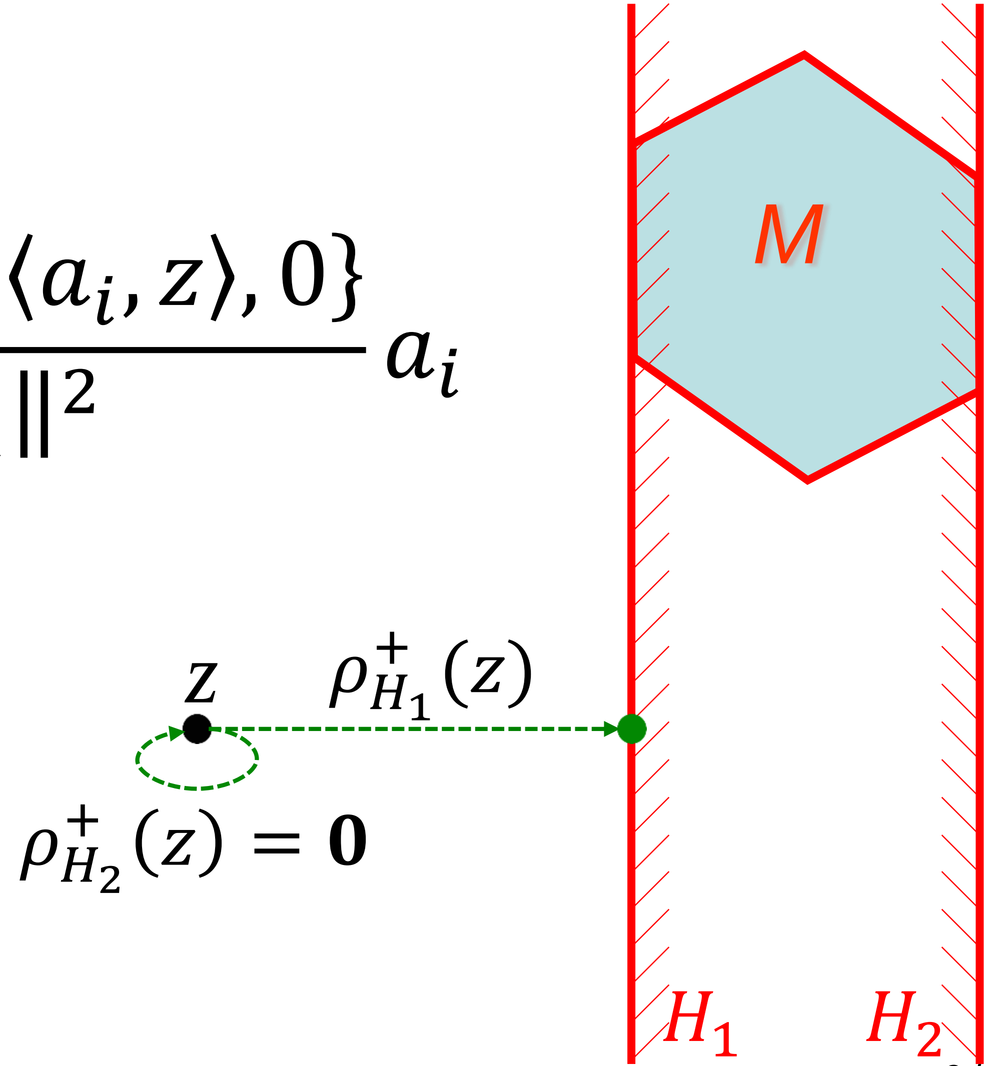
$$\rho_{H_i}(z) = \frac{b_i - \langle a_i, z \rangle}{\|a_i\|^2} a_i$$



$\|\cdot\|$ – Euclidean norm
$\langle a_i, z \rangle$ – dot product

Positive Slice of Projection Vector for Hyperplane H_i

$$\rho^+_{H_i}(z) = \frac{\min\{b_i - \langle a_i, z\rangle, 0\}}{\|a_i\|^2} a_i$$



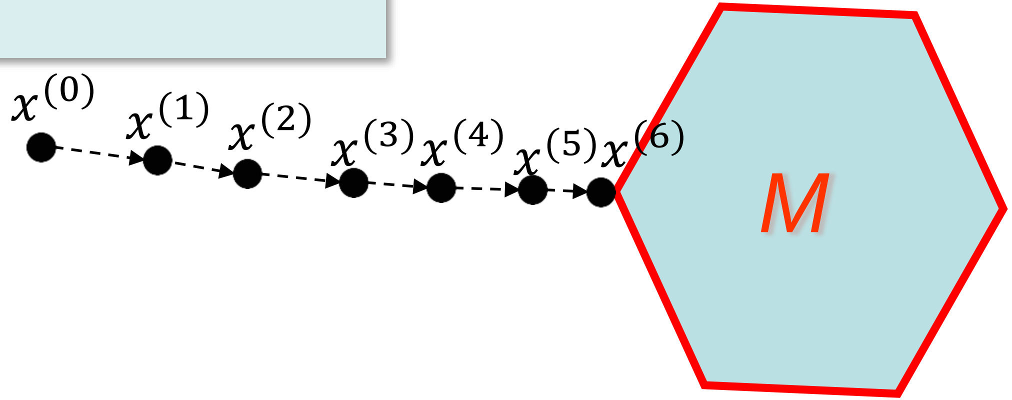
Projective Mapping

$$\varphi(x) = \frac{1}{h}\sum_{i=1}^{m} \rho_{H_i}^{+}(x)$$

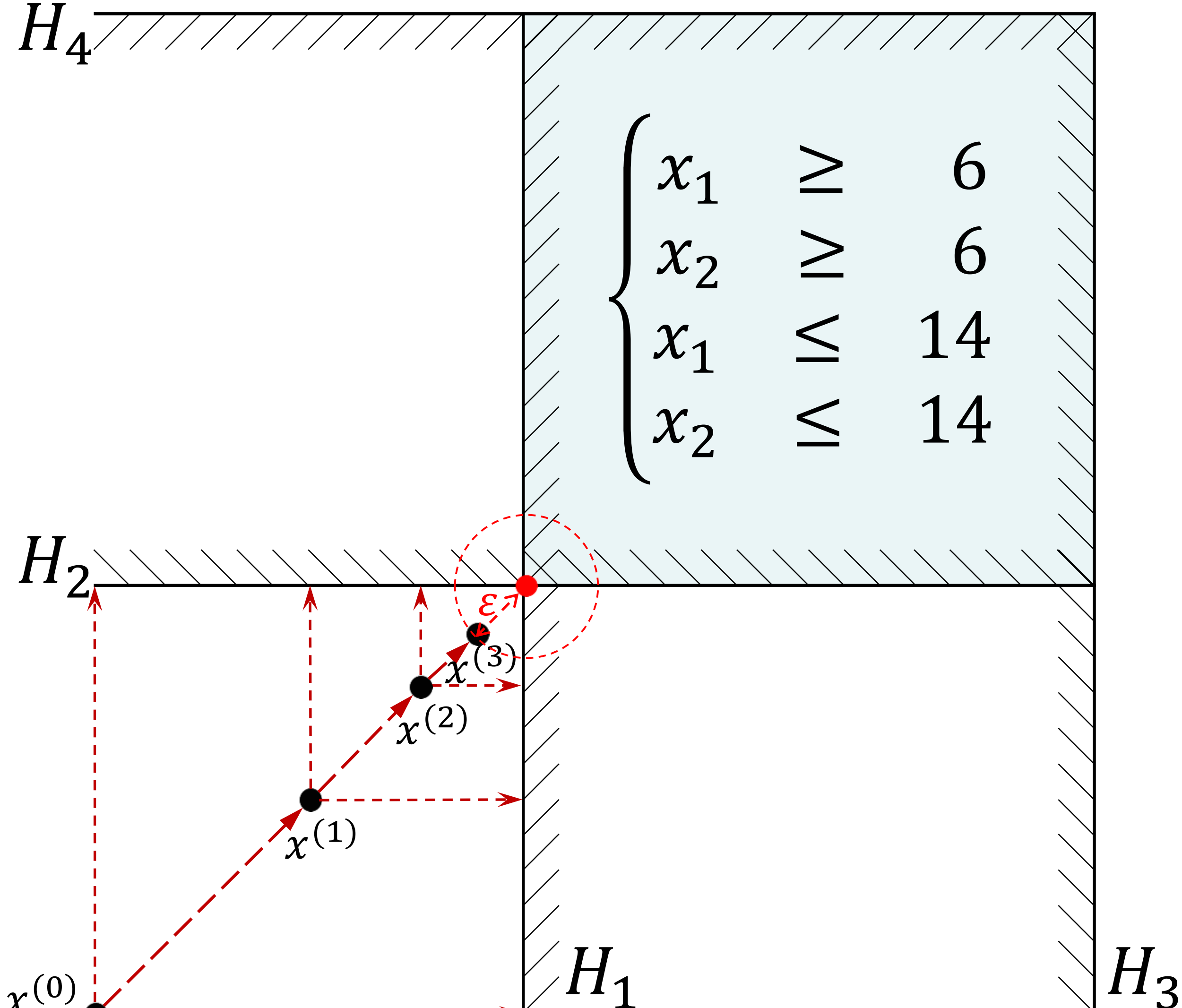
h – the number of nonzero terms in the sum $\sum_{i=1}^{m} \rho_{H_i}^{+}(x)$

Cimmino Iterative Algorithm for Inequalities

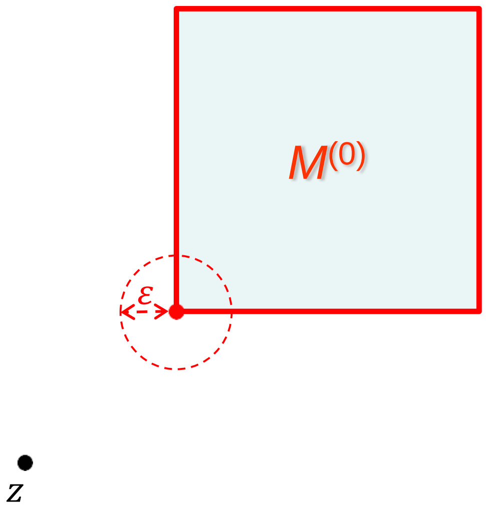
1. $x^{(0)} := \mathbf{0}$
2. $k := 0$
3. $x^{(k+1)} := x^{(k)} + \varphi^{(k)}\left(x^{(k)}\right)$
4. **if** $\left\| x^{(k+1)} - x^{(k)} \right\|^2 < \varepsilon^2$ **goto** 7
5. $k := k + 1$
6. **goto** 3
7. **stop**



How Algorithm works



Cimmino Algorithm is not suitable for Non-stationary Case



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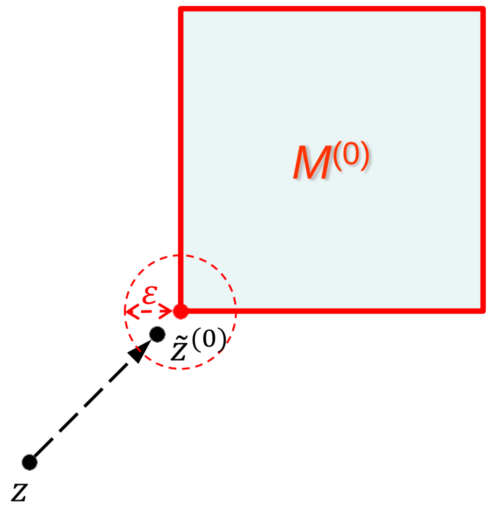

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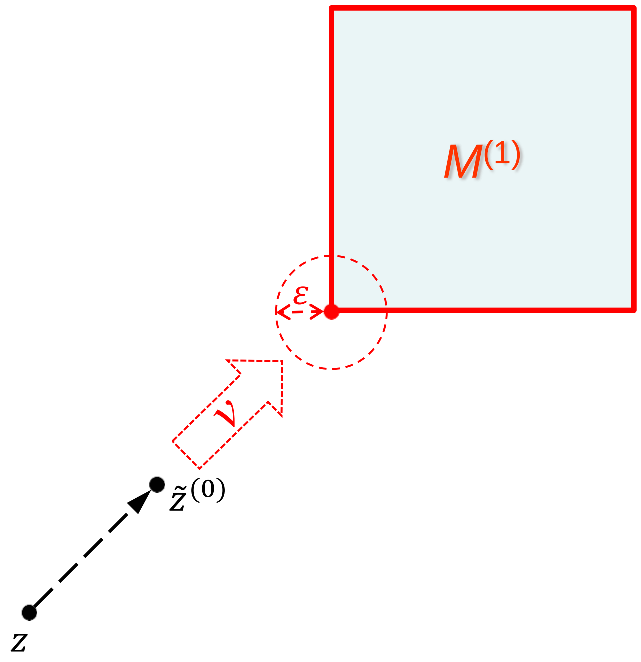

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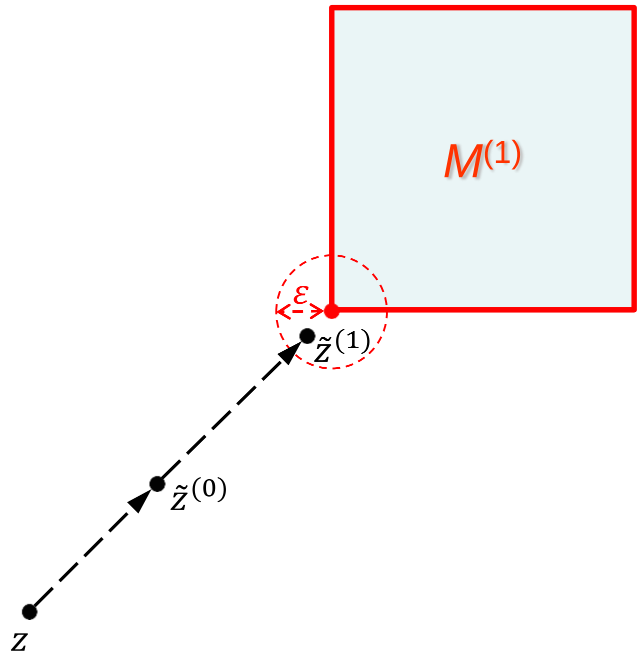

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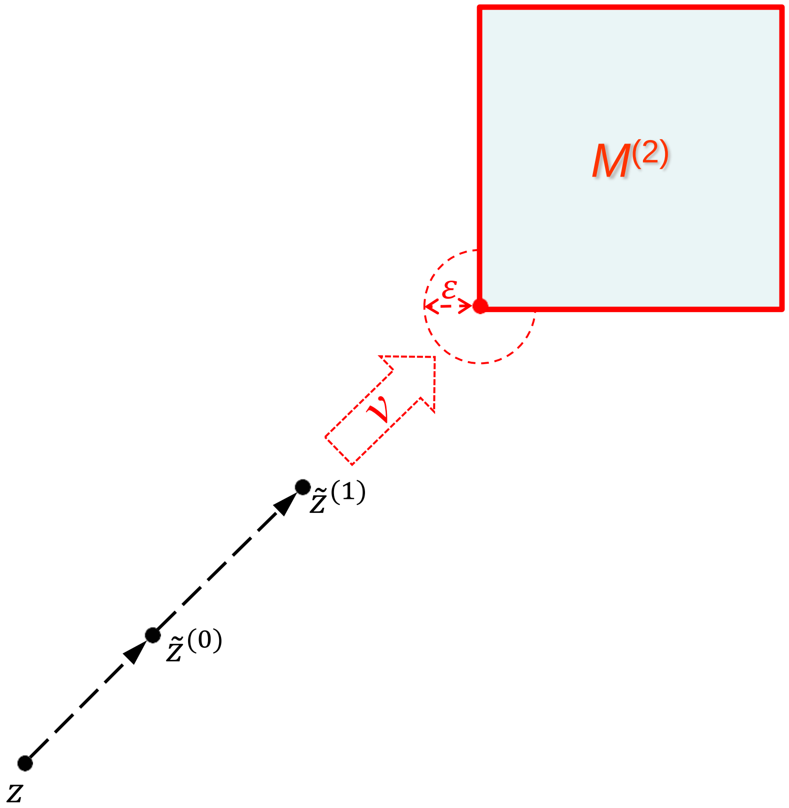

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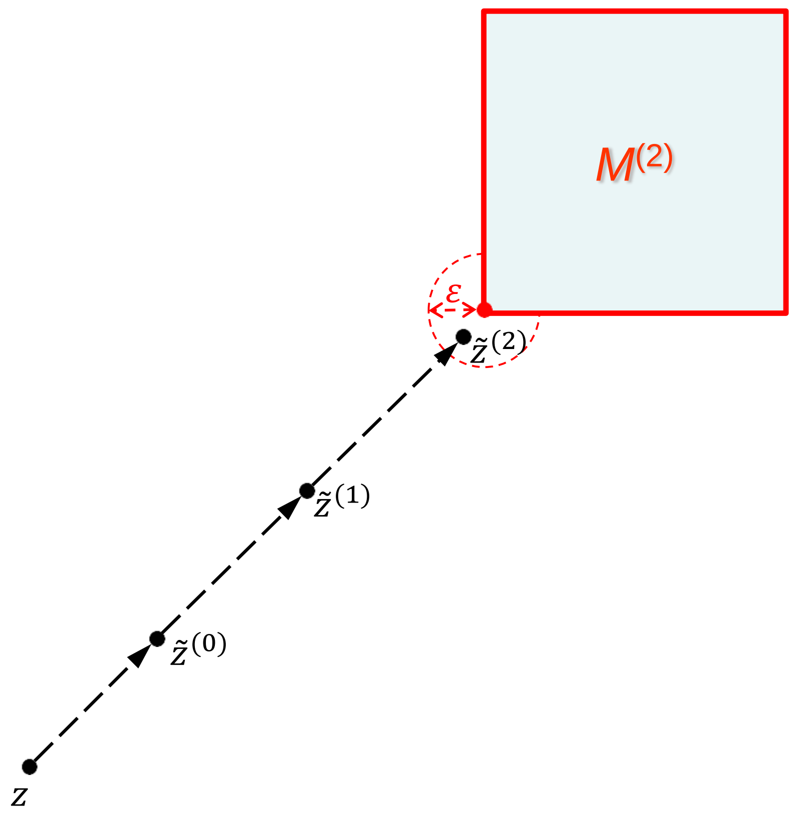

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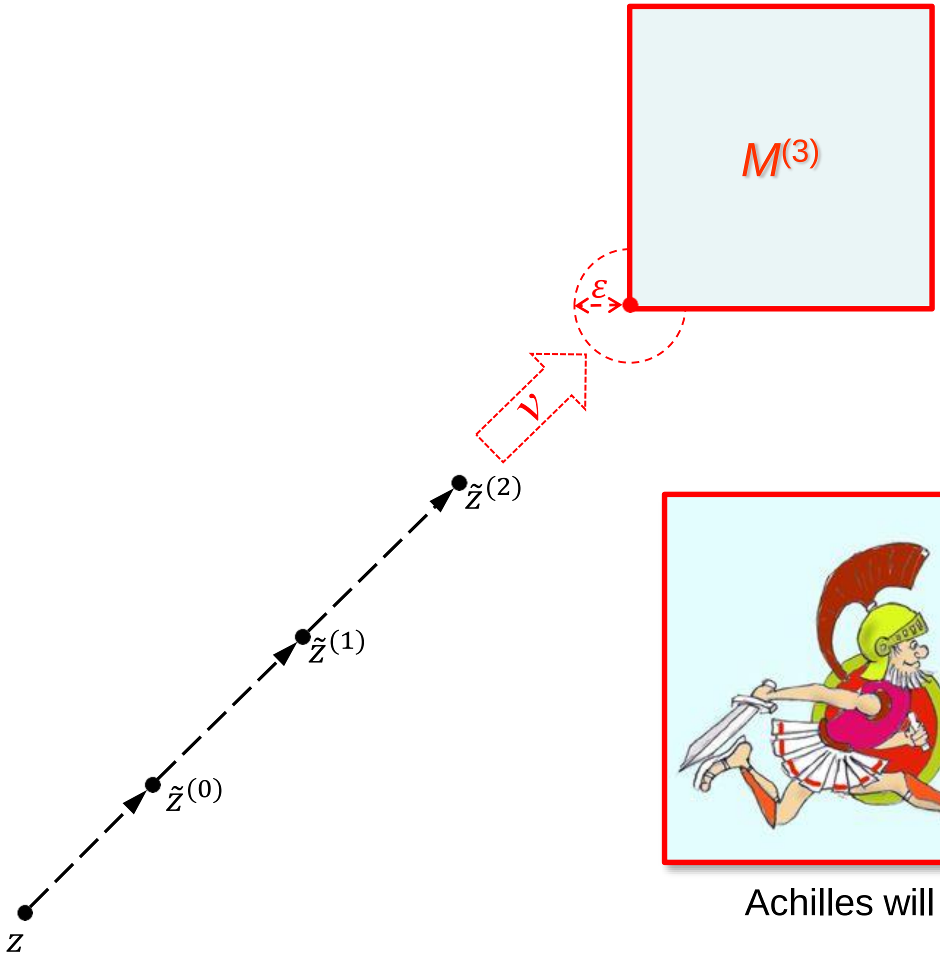
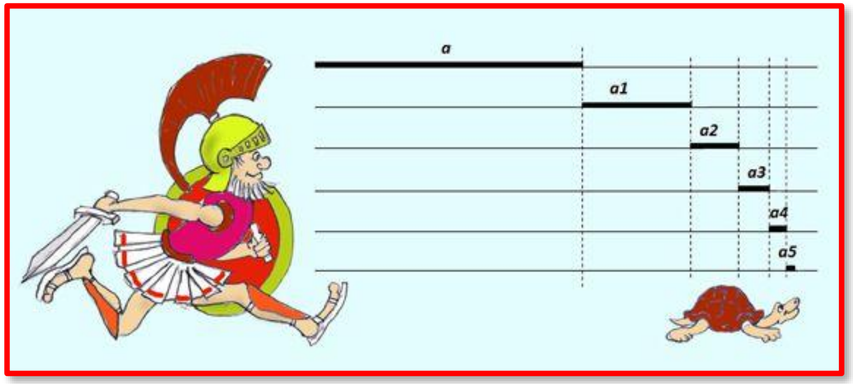




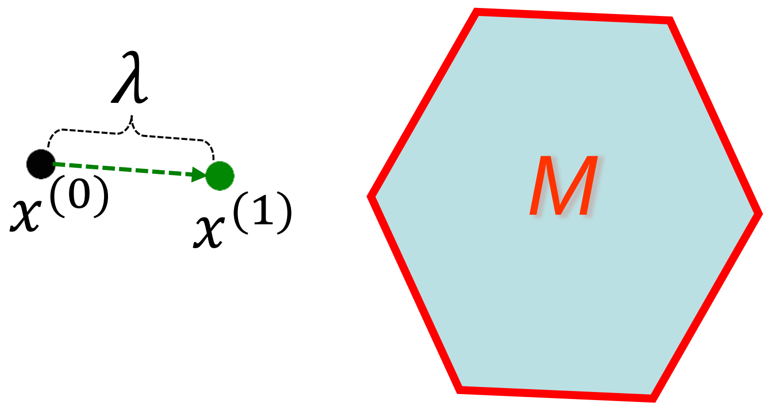
Achilles will never overtake the tortoise!

Modification of Projective Mapping

$$\varphi(x) = \frac{1}{h}\sum_{i=1}^{m} \rho^{+}_{H_i}(x) \quad \Rightarrow \quad \psi(x) = \lambda \frac{\varphi(x)}{\|\varphi(x)\|}$$

$$\lambda > 0$$

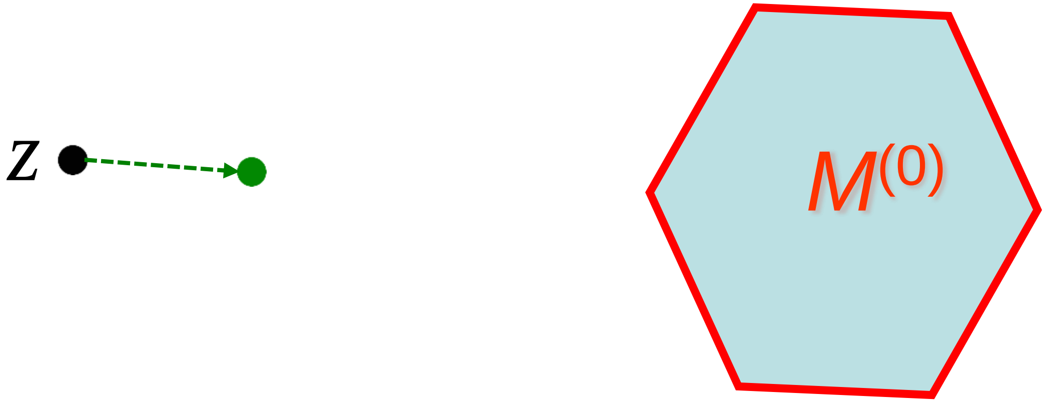
$$x^{(1)} = x^{(0)} + \psi\left(x^{(0)}\right)$$



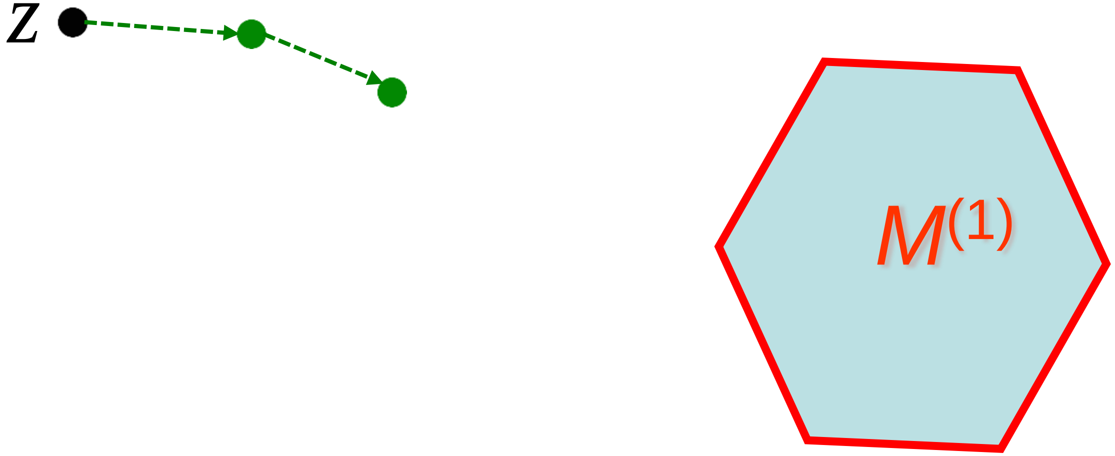
Modified Algorithm

1. $x^{(0)} := \mathbf{0}$
2. $k := 0$
3. $x^{(k+1)} := x^{(k)} + \psi^{(k)}\left(x^{(k)}\right)$
4. **if** $x^{(k+1)} \in M^{(k)}$ **goto** 7
5. $k := k + 1$
6. **goto** 3
7. **stop**

Operating of Modified Algorithm



Operating of Modified Algorithm



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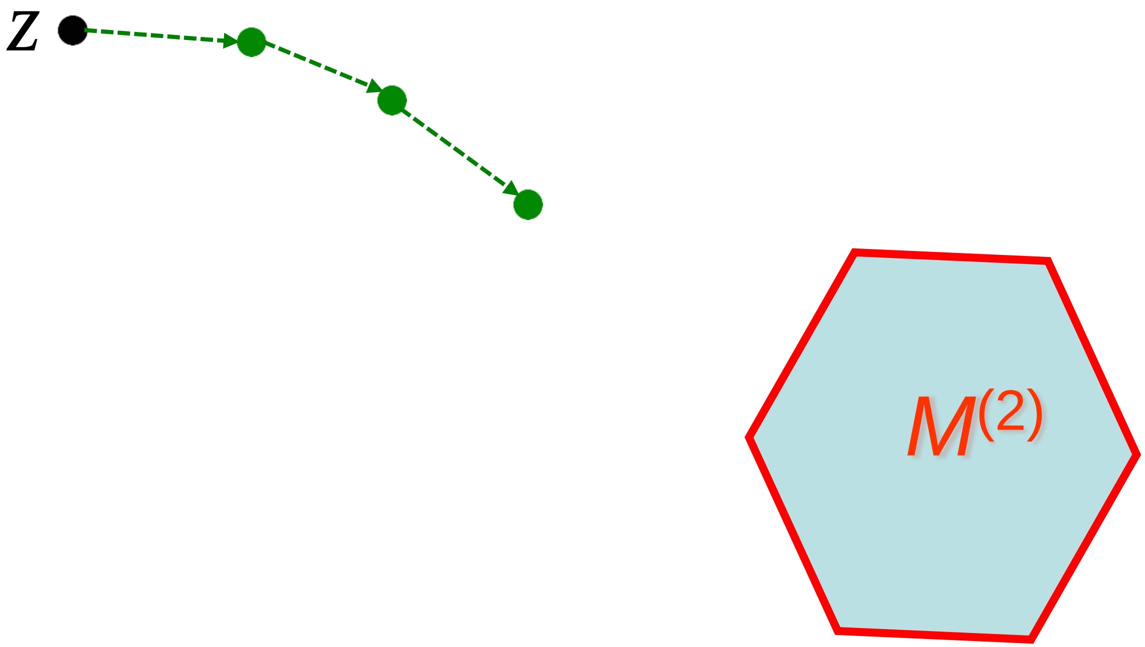

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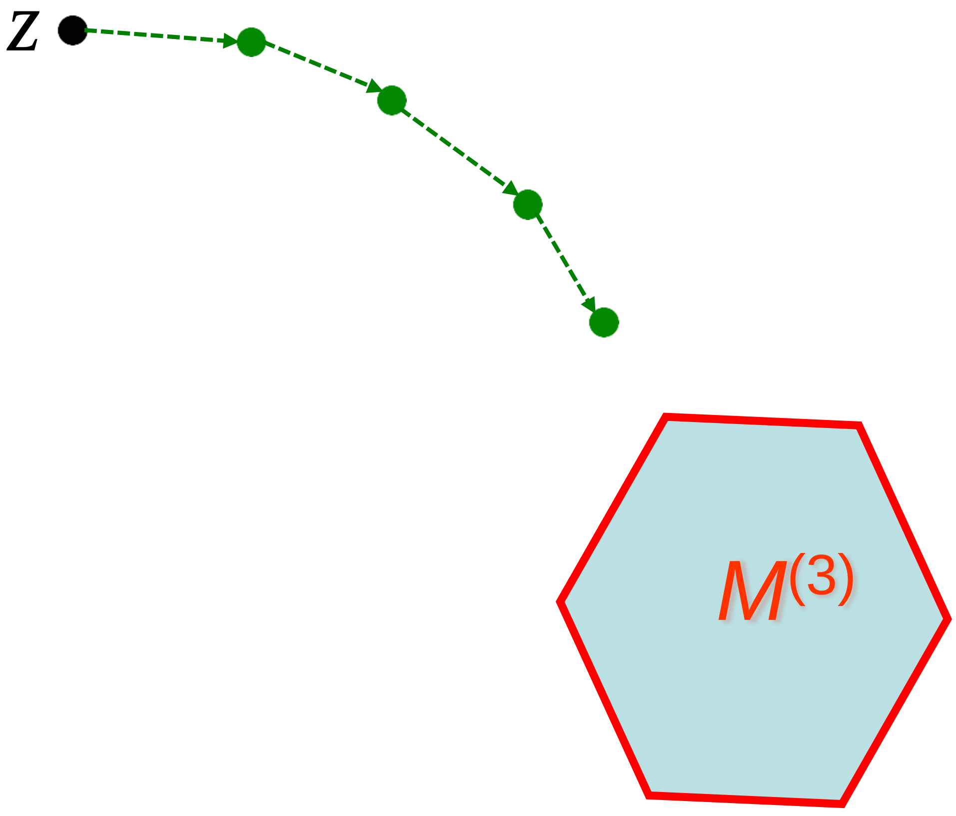

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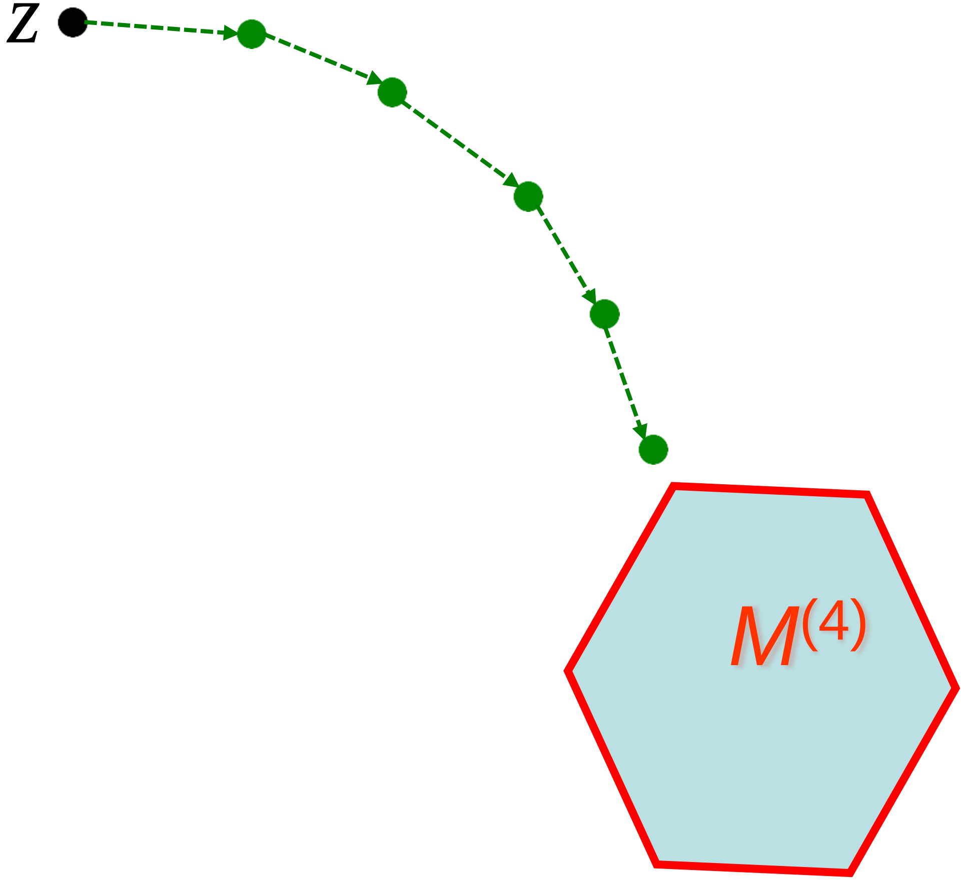

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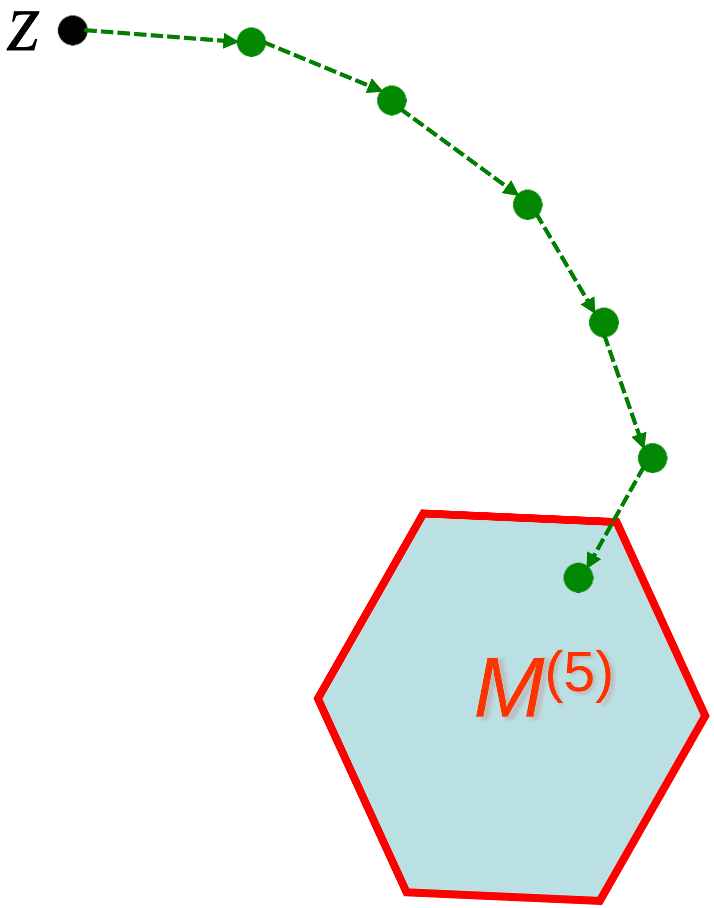


Achilles overtook the tortoise!

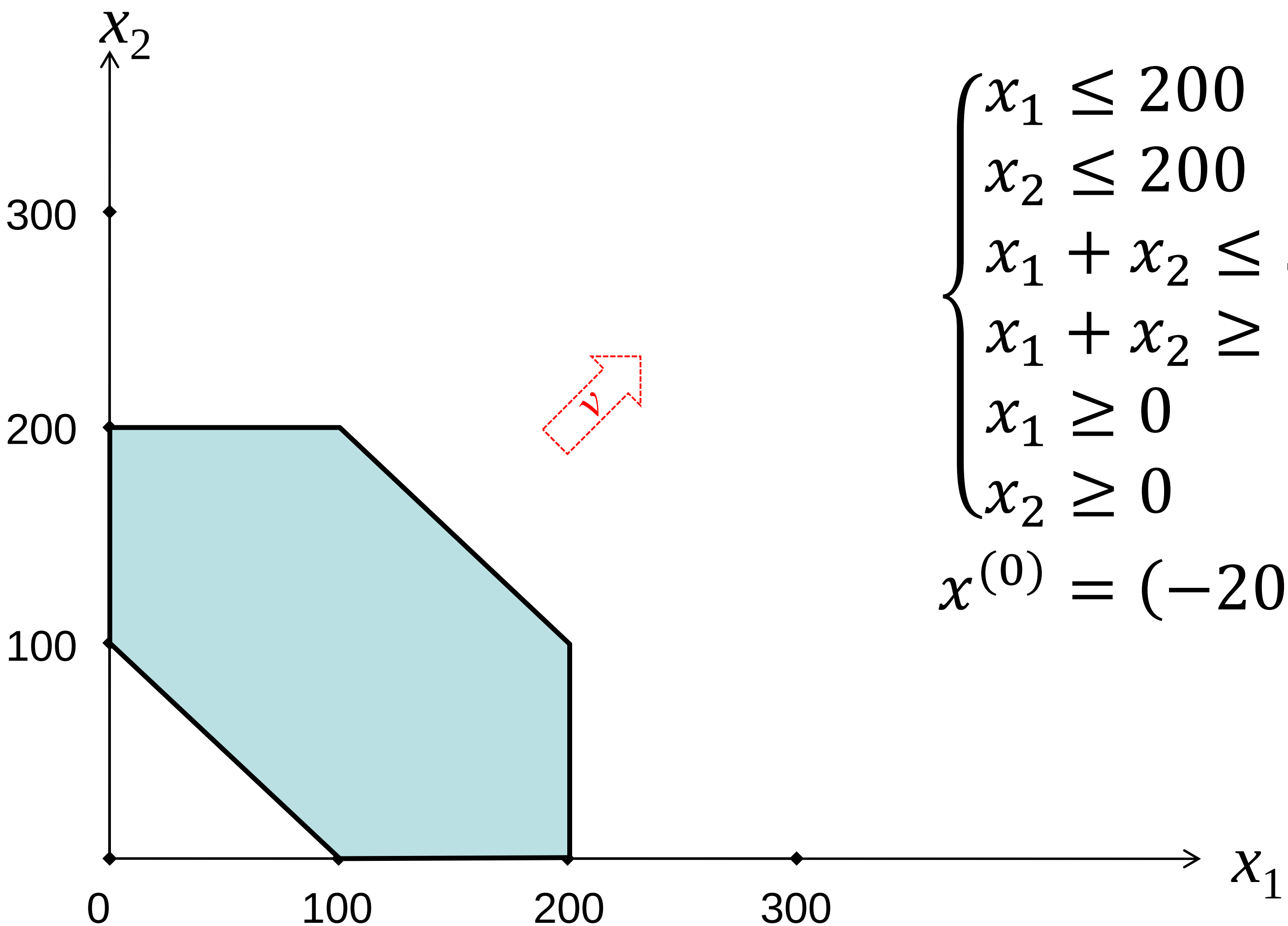
Synthetic Problem

$$
\left\{
\begin{array}{rcccccl}
x_0 & & & & & \leq & 200 \\
 & & x_1 & & & \leq & 200 \\
 & & & \ddots & & \cdots & \cdots \\
 & & & & x_{n-1} & \leq & 200 \\
x_0 & + & x_1 & \cdots & + \; x_{n-1} & \leq & 200(n-1)+100 \\
x_0 & + & x_1 & \cdots & + \; x_{n-1} & \leq & -100 \\
-x_0 & & & & & \leq & 0 \\
 & & -x_1 & & & \leq & 0 \\
 & & & \ddots & & \cdots & \cdots \\
 & & & & -x_{n-1} & \leq & 0
\end{array}
\right.
$$

Number of variables: n

Number of inequalities: $m = 2n + 2$

Synthetic Problem with n=2



$$\begin{cases} x_1 \leq 200 \\ x_2 \leq 200 \\ x_1 + x_2 \leq 300 \\ x_1 + x_2 \geq 100 \\ x_1 \geq 0 \\ x_2 \geq 0 \end{cases}$$

$$x^{(0)} = (-200, -200)$$

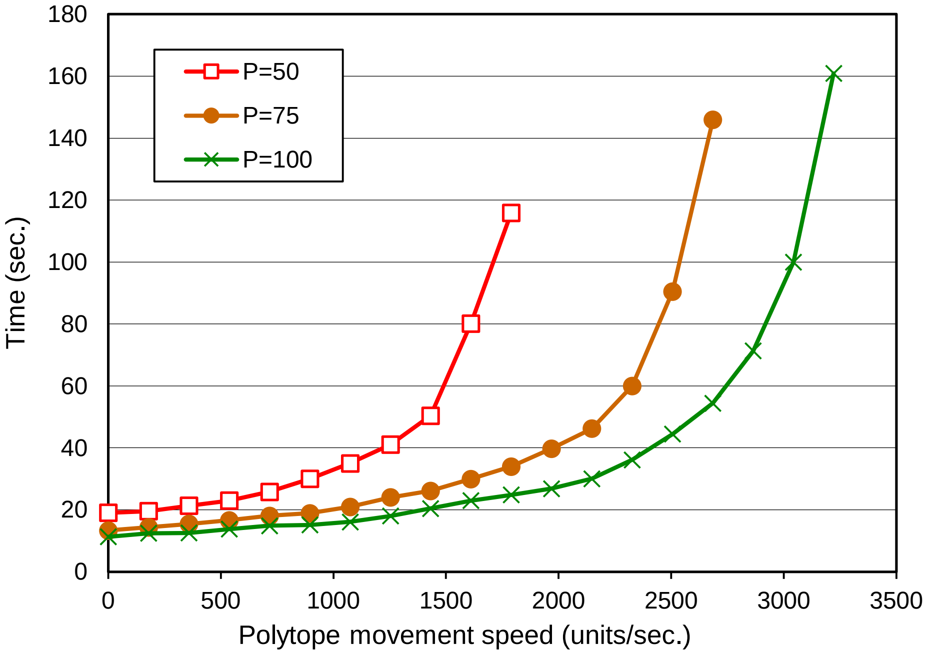
Supercomputer "Tornado SUSU"

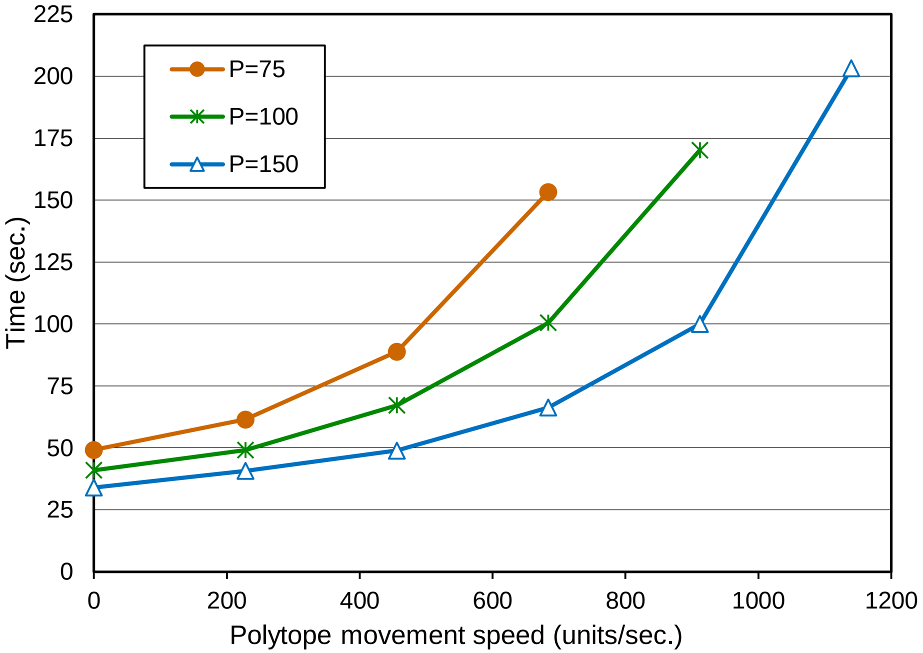
Number of nodes:	384
CPU:	2 x Intel Xeon X5680
RAM per node:	24 GB
Coprocessor:	Intel Xeon Phi SE10X:
Coprocessor memory:	8 GB
Computational network:	InfiniBand QDR
Control network:	Gigabit Ethernet
Operating system:	Linux CentOS 6.2

Computational Experiments

P – number of processor nodes

$$\lambda = 140$$





Number of variables: 32 000
Number of inequalities: 64 002

Number of variables: 54 000
Number of inequalities: 108 002

Thanks for your attention!